COVERING SPACES OVER CLASPERED KNOTS

ANDREW KRICKER

ABSTRACT. In this note we reconsider a familiar result in Vassiliev knot theory - that the coefficients of the Alexander-Conway polynomial determine the top row of the Kontsevich integral - from the point of view of Kazuo Habiro's clasper theory. We observe that in this setting the calculation reflects the topology of the universal cyclic covering space of a claspered knot's complement.

1. INTRODUCTION

In recent times the focus of the finite type theory of knots has shifted to questions of what of the topology of knots finite type invariants actually detect [Hab1, Hab2, KL, N, S]. In this respect, it seems that Kazuo Habiro's clasper theory will provide a powerful tool. This note will show how clasper theory reveals topological content in a previously combinatorial understanding: that the Alexander-Conway polynomial determines the top row of the Kontsevich integral [C, K, KSA, V], following the basic framework of [BNG]. The original conjecture in whose shadow this work lies is due to Melvin and Morton [MM].

We note that certain aspects of Habiro's clasper theory have been independently considered by Goussarov. In particular see the paper [G] and the web page [GW].

Take the following representation of the Alexander-Conway polynomial (as a formal power series in the variable h):

$$
\begin{aligned}
C_{K^+}(h) - C_{K^-}(h) &= (e^{\frac{h}{2}} - e^{-\frac{h}{2}})C_{K^s}(h), \\
C_U(h) &= 1.
\end{aligned}
$$

In the usual way K^+, K^- and K^s are three knots differing in a ball as an overcrossing, an undercrossing, and an orientation-preserving smoothing of the crossing, and U is the unknot. Take the coefficients of this series $C_K(h) = \sum_{n=0}^{\infty} c_{2n}(K)h^{2n}$. The coefficient c_{2n+1} vanishes and the coeffiient c_{2n} is finite type of order $2n$ [BN1]. Let the coefficients determine rational knot invariants $d_{2n}(K)$:

$$
(1.1) \qquad C_K(h) = \exp\left(-2\sum_{n=1}^{\infty} d_{2n}(K)h^{2n}\right).
$$

In these terms the top row of the Kontsevich integral is determined:

$$
(1.2) \qquad Z(K) = \exp\left(\frac{f(K)}{2}\theta + \sum_{n=1}^{\infty} d_{2n}(K)\omega_{2n} + X\right),
$$

where Z is the framed Kontsevich integral [LM] with the normalisation employed in [BN2] (so that $Z(U) = 1$), where θ is the web diagram with a single chord, where ω_{2n} is the "wheel with $2n$ spokes",

$$(1.3) \qquad \omega_2 = \vcenter{\hbox{}} \ , \ \omega_4 = \vcenter{\hbox{}} \ , \ \omega_6 = \vcenter{\hbox{}} \ , \ \dots$$

and where X is a series of diagrams with connected dashed graphs of negative Euler characteristic. (The "Chinese character diagram" wheels first appeared in [CV]. The papers [GH] and [W] contain this formula.)

This understanding follows from the theory of the Kontsevich integral, observations concerning the structure of the primitive subspace of the chord diagram algebra [CV], and the following theorem, where the weight system of c_{2n} is denoted $W(c_{2n})$. Pieces of the following theorem appear in many papers [BNG, C, K, KSA, V], most significantly [BNG]; [C] and [V] contain clear presentations of the final product:

1.1. Theorem.

$$(1.4) \qquad W(c_{2n})(Z_{2n}(K)) = c_{2n}(K),$$

$$(1.5) \qquad W(c_{m_1+m_2})(D_1 \# D_2) = W(c_{m_1})(D_1)W(c_{m_2})(D_2),$$

$$(1.6) \qquad W(c_{2n})(w_{2n}) = -2,$$

$$(1.7) \qquad W(c_{2n})(D) = 0 \quad \begin{array}{l} \textit{if the dashed graph of } D \\ \textit{has -ve Euler characteristic,} \end{array}$$

where Z_{2n} is the degree $2n$ term of Z, $D_i \in A_{m_i}(S^1)$ and D is a diagram in $Prim(A_{2n}(S^1))$.

A major virtue of clasper theory is that it constructs directly from some primitive web diagram D, a surgery presentation of a knot K which has the property that $Z(K) = 1 + D +$ higher order terms. Weight systems of invariants can thus be calculated on primitive web diagrams by evaluating the difference between the value the invariant takes on a single knot claspered from the unknot according to the diagram D and the value it takes on the unknot (as opposed to the usual double sum that comes from STU resolutions of trivalent vertices of the dashed graph followed by desingularisations of double points).

In this note the Alexander-Conway invariants of knots obtained by claspering the unknot will be calculated by explicitly constructing surgery descriptions of the universal cyclic covers of their complements. The knots associated to wheels are familiar as twisted variants of Ng's wheels [N] (and in fact, the approach presented in this work owes a debt to [N]). In this framework the vanishing theorem (Equation 1.7) recieves a topological explanation. This approach provokes some questions.

Generalisations of some of these observations in the direction of knot invariants which factor through Seifert matrices are contained in recent work due to Murakami and Ohtsuki [MO]. There are close relations with some extensive recent work due to Habiro, Kanenobu and Shima concerning finite-type invariants of ribbon 2-knots [HKS, HS].

2. Finite type invariants and clasper theory

The finite type theory of knots is based on a filtration of the $\mathbb{Q}$-vector space of finite, $\mathbb{Q}$-linear combinations of knots, K. Here this filtration will be introduced by means of *claspers*. Clasper theory is due to Kazuo Habiro [Hab1, Hab2] (extensively generalising precursors due to Matveev [Ma] and Murakami–Nakanishi [MN]). We note that certain aspects of this theory were independently considered by Goussarov [G, GW]. A clasper is an embedding of a band-summed pair of annuli in the complement of the knot.

(2.1)
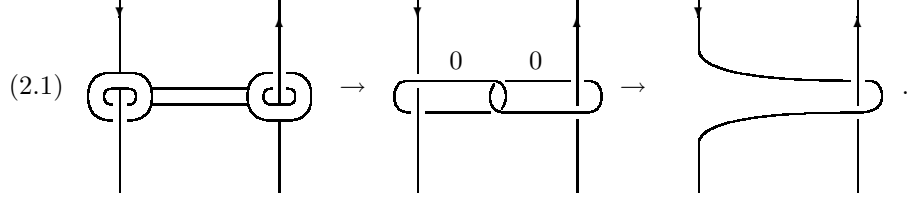
A clasper carries placement information for a surgery link. In the case of Figure 2.1 the surgery link is a 0-framed Hopf link, surgery on which effects a crossing change on the knot. Of course, two points on the knot may be "claspered" like this in many different ways corresponding to different homotopy classes of the band. Habiro has called a clasper of this sort a "quantum chord". In this work we will restrict ourselves to clasper decorations of knots such that each annulus is untwisted, and such that there is a diagram of the decoration (in the obvious sense) with each annulus linking the knot once as in Figure 2.1 (though we will soon expand this class to let annuli meet in something like a trivalent vertex) and with each band untwisted (immersed in the plane of the projection). This restriction essentially rules out half-twists of bands and is imposed here mainly to deal with sign issues in the isomorphisms presented below.

Decorate a knot K with n disjoint claspers $C = \{C_i\}$, as above. Write K^C for the knot obtained by surgery on such a set of claspers. Denote the following sum

$$[K, C] = \sum_{D \subset C} (-1)^{\#D} K^D \in \text{K}.$$

2.1. Definition. K_n is the subspace of K spanned by vectors $[K, C]$ where $\#C = n$.

The graded piece $\frac{\text{K}_n}{\text{K}_{n+1}}$ is isomorphic to $A'_n(S^1)$, the space of web diagrams of degree n [BN2]. $A'_n(S^1)$ is the $\mathbb{Q}$-vector space generated by web diagrams quotiented by the subspace described by STU and 1T relations. Web diagrams are graphs with univalent or trivalent vertices (such as the wheels of Equation 1.3), such that each trivalent vertex is equipped with a cyclic ordering of its incident edges and the univalent vertices are labelled (up to orientation-preserving diffeomorphism) with disjoint points on an oriented copy of S^1. The STU relations are

(2.2)

where the diagrams of the three terms differ only in the parts pictured. The 1T relations are spanned by diagrams with isolated chords.

Call this isomorphism $\phi_n : A'_n(S^1) \rightarrow \dfrac{K_n}{K_{n+1}}$. We will introduce Habiro's construction of this isomorphism in the next section, using clasper graphs.

2.2. **Clasper graphs.** Clasper theory has provided a tool to describe ϕ_n directly on web diagrams using knots decorated with *clasper graphs*. In a clasper graph, we extend the class of clasper decorations to allow annuli to meet in a set of Borromean rings, as below (retaining the requirement that there be a diagram such that no bands have half-twists):

(2.3)

.
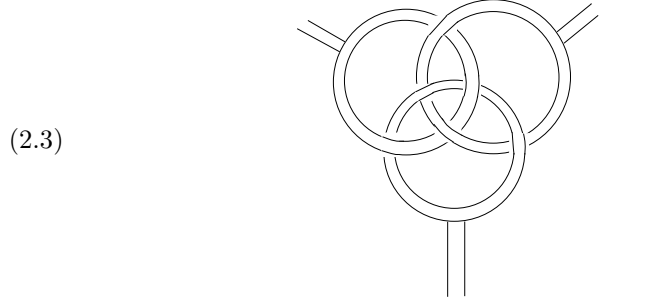
The annuli which link the knot are called the *leaves* of the clasper graph. Clasper theory covers a much more extensive class of objects than the class employed in our calculation, and we enthusiastically direct the reader to [Hab1, Hab2].

Such a graph as we have described here has an underlying web diagram obtained by replacing the Borromean rings with trivalent vertices (oriented according to the diagram in which bands have no half-twists) and leaves with univalent vertices appropriately ordered on the S^1, though of course many different clasper graphs will have the same underlying diagram. The degree of a clasper graph is the degree of the underlying diagram (so is half the sum of the numbers of leaves and sets of Borromean rings in the graph).

Decorate a knot K with some degree n clasper graph $G = \{G_1, \dots, G_k\}$ having underlying web diagram D, where the clasper subgraphs $\{G_1, \dots, G_k\}$ correspond to the connected components of the dashed graph of D.

2.3. Theorem ([Hab2])**.**

(2.4) $$\phi_n(D) = (-1)^{l+deg(G)} \sum_{S \subset G} (-1)^{\#G - \#S} K^S,$$

where l is the number of leaves of G, the sum is over all (2^k) subsets of the set of clasper subgraphs corresponding to the connected components of the dashed graph of the underlying diagram of G, and $\#S$ is the cardinality of S.

Observe that if the underlying web diagram has connected dashed graph (so that it is primitive) then its image in $\dfrac{K_n}{K_{n+1}}$ is a difference $(-1)^{l+n}(K^G - K)$.

2.4. **Remark.** There is a technical observation from clasper theory that will be required in the sequel. Consider a knot obtained from some other knot by surgery on a clasper graph that is of the sort described above, except that one clasper has

a free end, as below:

(2.5)
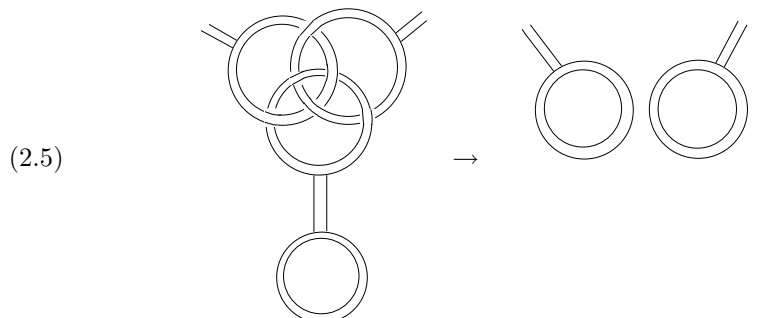
It is clear that a single pair of surgeries on the components associated with the clasper with a free end will leave the graph as shown. Continuing, it is obvious that a component with a free end may be removed altogether without affecting the resultant knot. (This is related to how one sees that surgery on a clasper graph leaves the ambient manifold invariant.)

2.5. **Weight systems.** The weight system of a degree n finite type invariant V (so that $V(K_{n+1}) = 0$) is the composition $W(V) = V \circ \phi_n$.

In this note it will be more convenient to use the space of web diagrams quotiented by only the STU relations: it will be denoted here $A_n(S^1)$. The clasper graph construction associates to any knot invariant of finite type of degree n a dual vector to $A_n(S^1)$. This dual vector satisfies 1T relations and thus projects to the corresponding weight system. The weight system stated in the main theorem is precisely the appropriate dual vector to $A_n(S^1)$ that projects to the weight system for c_{2n}. This presentation is required here as $A_n(S^1)$ is the range of the version of the Kontsevich integral referred to in the introduction.

3. COVERING SPACES OF CLASPERED KNOTS

3.1. **The Alexander polynomial.** A short review is in order. A standard reference is [Rolf]. Take $\widetilde{S^3 \setminus K}$, the covering space of $S^3 \setminus K$ corresponding to the abelianisation of $\pi = \pi_1(S^3 \setminus K)$ (that is, to the short exact sequence $0 \to [\pi, \pi] \to \pi \to \mathbb{Z} \to 0$), with group of deck transformations $\mathbb{Z}$ generated by some T. This is called the universal cyclic covering space. Defining a $\mathbb{Z}[t, t^{-1}]$ action on $H_1(\widetilde{S^3 \setminus K}, \mathbb{Z})$ by $\sum_i a_i t^{n_i}.h = \sum_i a_i (T_*)^{n_i}(h)$, $H_1(\widetilde{S^3 \setminus K}, \mathbb{Z})$ becomes a $\mathbb{Z}[t, t^{-1}]$-module. The Alexander polynomial of K (up to multiplication by units) is a generator of the order ideal of $H_1(\widetilde{S^3 \setminus K}, \mathbb{Z})$.

3.2. **Complete diagrams.** Let us call a diagram *complete* if the trivalent vertices of its dashed graph have at most one incident edge whose other end is a univalent vertex (this extends a term due to Ng). A primitive web diagram is equal to a sum of complete primitive web diagrams of lesser or equal Euler characteristic. Thus we may restrict out attention to complete diagrams in the proof of Theorem 1.1.

3.3. **Universal cyclic covering spaces.** Take a complete primitive web diagram $D \in A_n(S^1)$. Decorate the unknot with a clasper graph G with underlying diagram D: so that $(-1)^{l+n}(U^G - U)$ represents $\phi_n(D)$ in $\frac{K_n}{K_{n+1}}$. Choose G so that if the leaves are removed then the resulting graph is unlinked from the knot (this can

always be done, and is done here essentially to simplify the choice of Seifert surface in the exposition below). $W_n(c_n)(D)$ is equal to $(-1)^{l+n}c_n(U^G)$. We calculate this by building a surgery description of the universal cyclic cover of $S^3 \setminus U^G$.

Observe that near a leaf of the clasper graph, a single surgery on the clasper ending in the leaf brings the picture to Figure 3.1.

(3.1)
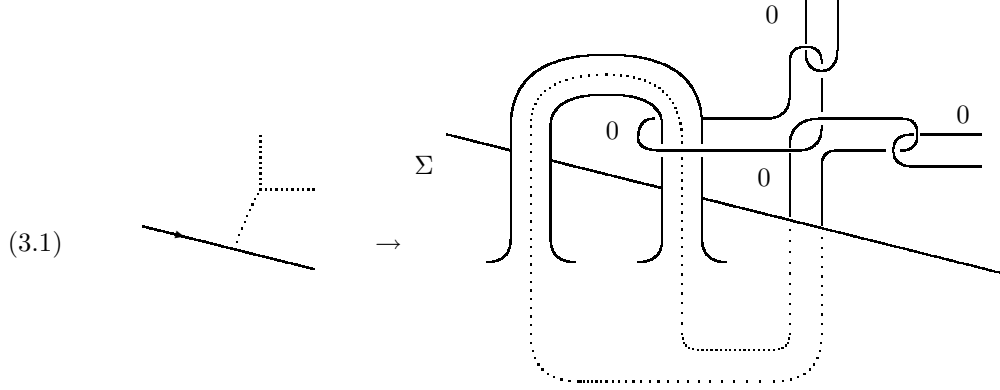
In Figure 3.1 a Seifert surface Σ for the unknot which is disjoint from the surgery link has been chosen. Use this surface to construct the universal cyclic cover of the unknot in the usual way. [In outline: we "split" $S^3 \setminus U$ along $\Sigma \setminus \partial\Sigma$ (which we will denote $°\Sigma$) leaving a manifold X with boundary $°\Sigma^{\pm}$, two copies of $°\Sigma$ which meet along U (although U is excluded from the space). $S^3 \setminus U$ is recovered from an identification $\sigma :°\Sigma^+ \to°\Sigma^-$. Take $\mathbb{Z}$ copies of the triple $(°\Sigma^-, X, °\Sigma^+)$ denoting them $(°\Sigma_i^-, X_i, °\Sigma_i^+)$ and $\mathbb{Z}$ of the appropriate maps $°\Sigma_i^+ \to°\Sigma_{i+1}^-$. $\widetilde{S^3 \setminus U}$ is recovered as the identification space of this system. A rather more rigorous discussion of this procedure may be found in [Rolf]].

Denote the covering map $p : \widetilde{S^3 \setminus U} \to S^3 \setminus U$. As Σ is disjoint from the surgery link it is clear that L lifts, $p^{-1}(L)$, to a surgery presentation for $\widetilde{S^3 \setminus U^G}$.

3.4. **Proof of the vanishing theorem, Equation** 1.7. Take G some graph clasper for U with underlying diagram some chosen complete primitive diagram D of negative Euler characteristic. The dashed graph of such a diagram will have at least one trivalent vertex which has no incident edges connecting univalent vertices. Thus the corresponding surgery link will have at least one set of three components linking in a set of Borromean rings in X_0, lifted to all the X_i. But the Alexander polynomial is calculated from the homology of $\widetilde{S^3 \setminus U^G}$ and this coincides with the homology of $\widetilde{S^3 \setminus U^{G'}}$ where G' is obtained from G by unlinking that set of

Borromean links:

(3.2)
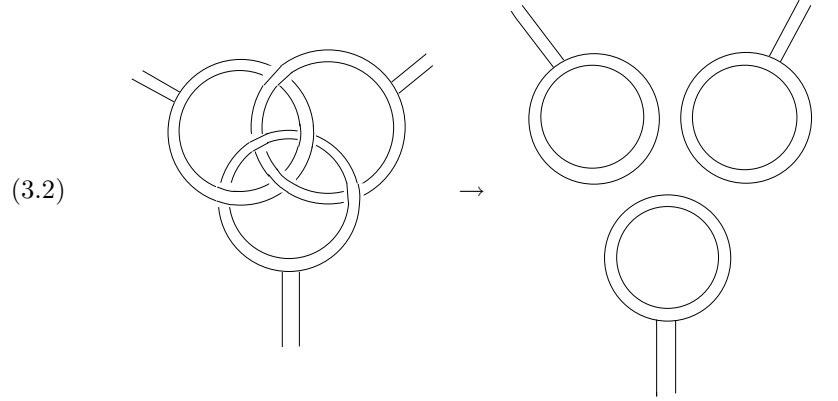
Further, the observation in Remark 2.4 shows that $U^{G'}$ is in fact the unknot, so that U^G has trivial Alexander polynomial.

3.5. **Wheels.** We turn our attention to the wheels. Denote by W_{2n} the graph clasper for the unknot with underlying graph corresponding to the planar embedding of the wheel ω_{2n}. For example, (U, W_2) is

(3.3)
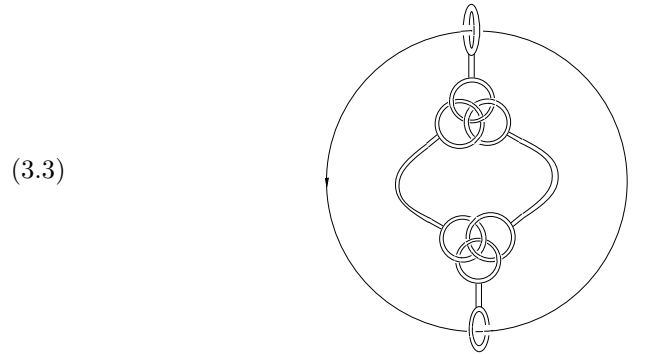
This sequence represents the wheel diagrams in the associated graded vector space: $\phi_{2n}(\omega_{2n}) = U^{W_{2n}} - U$.

Number the leaves of W_{2n} clockwise $1 \ldots 2n$. Around leaf i of the clasper graph the picture looks as follows (where $i + 1(2n)$ means $i + 1$ modulo $2n$, etc):

(3.4)
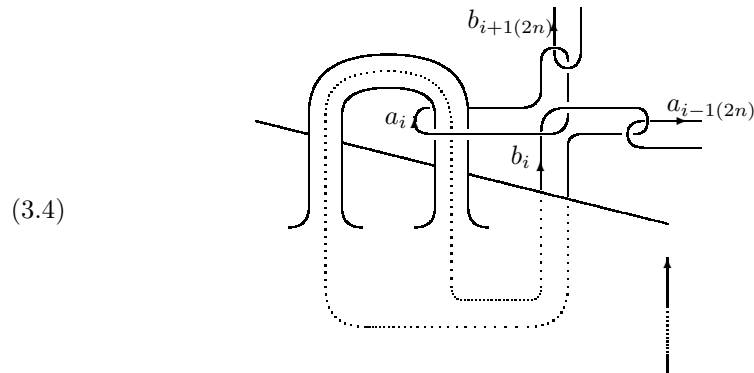
Take the universal cyclic cover of $S^3 \setminus U^{W_{2n}}$ by way of the surface Σ as in Subsection 3.3. Build the covering space so that in the figure, if the projection of a path starting in X_i travels through the surface in the direction indicated then the path is now in X_{i+1}. Choose the generating covering transformation as the one which maps X_i to X_{i+1}.

We calculate the $\mathbb{Z}[t, t^{-1}]$-module $H_1(\widetilde{S^3 \setminus U^{W_{2n}}}, \mathbb{Z})$. $H_1(\widetilde{S^3 \setminus U^{W_{2n}}}, \mathbb{Z})$ has a generator for each component of the surgery link. Thus it has a generator for each lift of the a_i and the b_j. Choose meridians α_i and β_j having linking number $+1$ with the associated components as oriented in the figure to represent these generators.

Relations are obtained by writing the lifts of the (0-framed) longitudes of the a_i and b_j in terms of these meridians. Up to translation there are $4n$ relations:

$$\text{(3.5)} \qquad \begin{aligned} \text{Around } \alpha_i &: \quad (t^{-1} - 1)\beta_i + \beta_{i+1(2n)} = 0, \\ \text{Around } \beta_i &: \quad (t - 1)\alpha_i + \alpha_{i-1(2n)} = 0, \end{aligned}$$

where i runs from 1 to $2n$ and in some cases has been written mod $2n$.

It follows from this that

$$\text{(3.6)} \qquad H_1(\widetilde{S^3 \setminus U^{W_{2n}}}, \mathbb{Z}) = \frac{\mathbb{Z}[t, t^{-1}]}{(1 - (1 - t)^{2n})} \oplus \frac{\mathbb{Z}[t, t^{-1}]}{(1 - (1 - t^{-1})^{2n})},$$

and that the corresponding invariants are

$$\text{(3.7)} \qquad A_{U^{W_{2n}}}(t) = (1 - (1 - t)^{2n})(1 - (1 - t^{-1})^{2n}),$$

$$\text{(3.8)} \qquad C_{U^{W_{2n}}}(h) = 1 - 2h^{2n} + \dots,$$

as is required for the proof of Equation 1.6.

The form of A above $(f(t)f(t^{-1}))$ is that expected for a slice knot, and in fact one can show that $U^{W_{2n}}$ is ribbon, in fact a twisted form of Ng's wheels [N].

4. Questions

Notice that the result for diagrams whose graphs have negative Euler characteristic is better than we could have expected. If G is a graph clasper for U with underlying complete, primitive web diagram D of degree n with negative Euler characteristic then all that was required was that $C(U^G) = 1 + o(h^{n+1})$. However we have found that $C(U^G) = 1$.

This prompts the following question, whose (positive) answer is part of a work in progress:

4.1. Question. *With terms as in the above paragraph, is it true that*

$$\text{(4.1)} \qquad Z(U^G) = 1 \ mod \ span_{\mathbb{Q}} \left\{ D' \ s.t. \ \chi(D') \le \chi(D) \right\},$$

where $\chi(D)$ is the Euler characteristic of the dashed graph of D?

Further observe that the branched cyclic cover over a knot obtained by claspering the unknot by a clasper graph, the dashed graph of whose underlying diagram has negative Euler characteristic, is a $\mathbb{Z}$-homology sphere. Does its n-triviality reflect the graph of its diagram in some way?

References

[BN1] Bar-Natan,D., *Weights of Feynman diagrams and the Vassiliev knot invariants*, preprint, (1991).

[BN2] Bar-Natan, D., *On the Vassiliev knot invariants*, Topology **34** (1995), 423–472.

[BNG] Bar-Natan, D. and Garoufalidis, S., *On the Melvin-Morton-Rozansky conjecture*, Invent. Math. **125** (1996), 103-133.

[C] Chmutov, S.V., *Combinatorial analog of the Melvin-Morton conjecture*, published in proceedings of Knots '96, World Scientific (1997).

[CV] Chmutov, S.V. and Varchenko, A.N., *Remarks on the Vassiliev knot invariants coming from sl2*, Topology **36** (1997), 153–178.

[GH] Garoufalidis, S. and Habegger, N., *The Alexander polynomial and finite type 3-manifold invariants*, to appear in Math. Annalen.

[G] Goussarov, M., *Knotted graphs and a geometrical technique of n-equivalence*, preprint circa 1995.

[GW] Papers of Goussarov M., `http://www.ma.huji.ac.il~drorbn/Goussarov`

[Hab1] Habiro, K., *Claspers and the Vassiliev skein modules*, preprint University of Tokyo (1997).

[Hab2] Habiro, K., *Clasper theory and finite type invariants of links*, Geometry and Topology **4** (2000), 1–83.

[HKS] Habiro, K. Kanenobu, T. and Shima, A., *Finite type invariants of ribbon 2-knots*, University of Tokyo preprint, (1998).

[HS] Habiro, K. and Shima, A., *Finite type invariants of ribbon 2-knots, II*, University of Tokyo preprint, (1998).

[KL] Kalfagianni, E. and Lin, X-S., *Regular Seifert surfaces and Vassiliev knot invariants*, preprint `math.GT/9804032`.

[K] Kricker, A., *Alexander-Conway limits of many Vassiliev weight systems*, J. Knot theory and its Rams. (1997), 327–358.

[KSA] Kricker, A., Spence, B. and Aitchison, I., *Cabling the Vassiliev invariants*, J. Knot theory and its Rams. (1997), 687–714.

[LM] Le, T.Q.T. and Murakami, J., *The universal Vassiliev-Kontsevich invariant for framed oriented links*, Comp. Math. **102** (1996), 42–64.

[Ma] Matveev. S., *Generalised surgery of 3-dimensional manifolds and representations of homology 3-spheres* (in Russian), Mat. Zametki, **42** (1987) 268-275

[MM] Melvin. R. M. and Morton H.R., *The coloured Jones function*, Comm. Math. Phys. **169** (1995) 501–520.

[N] Ng, K.Y., *Groups of ribbon knots*, Topology **37** (1998) 441–458.

[MN] Murakami, H. and Nakanishi, Y., *On a certain move generating link homology*, Math. Ann. **284**, (1989), 75–89.

[MO] Murakami, H., Ohtsuki, T., *Finite type invariants of knots via their Seifert matrices*, preprint (1998).

[Rolf] Rolfsen, D., *Knots and links*, Mathematics Lecture Series **7** (1976), Publish or Perish, Inc.

[S] Stanford, T., *Vassiliev invariants and knots modulo pure braid subgroups*, preprint `math.GT/9805092`.

[V] Vaintrob, A. *Melvin-Morton conjecture and primitive Feynman diagrams*, preprint `q-alg/9605028`.

[W] Willerton, S. *The Kontsevich integral and algebraic structures on the space of diagrams*, University of Melbourne preprint (1998).

DEPARTMENT OF MATHEMATICAL AND COMPUTING SCIENCE, TOKYO INSTITUTE OF TECHNOLOGY

E-mail address: `kricker@is.titech.ac.jp`